Transition to turbulence in the free convection boundary layers on an inclined heated plate

By D. J. TRITTON

Department of Aeronautical Engineering, Indian Institute of Science, Bangalore

(Received 11 September 1962)

An experimental study has been made of transition to turbulence in the free convective flows on a heated plate. Observations have been made with the plate vertical and inclined at angles up to about 50° to the vertical, both above and below the plate. A fibre anemometer was used to survey the region of intermittent turbulence. Information has thus been obtained about the range of Grashof numbers over which transition takes place. Even when the plate is vertical the region of intermittent turbulence is long. When it is inclined, this region becomes still longer in the flow below the plate as a result of the stabilizing stratification, a Richardson number effect. It is possible to have a whole flow such that it should be described as transitional, not laminar or turbulent. It was noticed that in this flow and the vertical plate one, the velocity during the laminar periods could be either of two characteristic values, one of them close to zero. The behaviour above an inclined plate could be interpreted largely as a trend towards the behaviour described in a preceding paper.

The observations on the stably stratified flow help to explain a feature of glacier winds.

1. Introduction

The general features of the free convective flow produced by a vertical heated flat plate are well known (Prandtl 1952, § V, 16; Squire 1953, § 20; Kraus 1955; Schlichting 1955, Chap. XIV). Provided the Grashof number is high enough—and in practice it always is—the velocity and temperature fields have a boundary-layer character. Consequently, the distance, x, from the lower edge governs the flow and, except very close to the upper edge, the plate may be regarded as semi-infinite. The appropriate non-dimensional representation of x is the Grashof number (introduced below for a more general case). The fluid velocity has to approach zero both at the plate and far from it and the velocity profile has both a maximum and a point of inflexion. These general features are, of course, common to laminar, transitional and turbulent flow.

We may anticipate that they apply also when the plate is inclined at an angle to the vertical. There is then a component of the buoyancy force normal to the plate. However, the way this acts on the flow is quite different from the action of the component parallel to the line of greatest slope (Prandtl 1952, p. 422; Tritton 1963 *b*, § 6). It gives rise to a stratification of the flow, and so a Richardson number effect (Prandtl 1952, pp. 381–4). This stratification has a stabilizing effect on the convection below a heated inclined plate, where hotter (and so lighter) air is above colder. Above the plate the situation is reversed and the stratification is destabilizing.

Thus, in comparing the flow at different angles, it is appropriate to make the comparison at positions that are influenced in the same way by the longitudinal buoyancy force, that is at the same values of the Grashof number based on $g \cos \alpha$

$$Gr = \frac{g \cos \alpha}{\nu^2} \frac{T_1 - T_0}{T_0} x^3$$

(the basic notation throughout this paper is that given in Tritton 1963*b*, § 2). This form of the Grashof number is only for a gas; we may as well make this restriction immediately as differences in Prandtl number, ν/κ, prevent results from being transferred quantitatively from one fluid to another, and the experiments to be reported here used air throughout.

This paper describes another aspect of the experimental work on this topic introduced in the preceding paper (Tritton 1963*b*), namely how the transition to turbulence changes in position and character as the inclination is varied. This is basic to a full investigation of this type of flow. It is interesting also for the light it throws on the effect of stratification on the transition of a shear flow. Experiments have been carried out on both the flow below the plate and that above, but the former make up the more significant part of this paper. Of course, the experiments included the case of a vertical plate for comparison purposes.

I know of no previous experimental work on transition in the inclined plate flow, though there has been an attempt at the hydrodynamic stability problem (Plapp 1957). There are several experimental estimates of the position at which turbulence sets in with a vertical heated plate. These will be compared with my own observations in § 9; for the moment we just note that the results are highly divergent.

Interpretation of transition data is helped by knowledge of the laminar and fully turbulent flows, and so the next two sections will consist of a few remarks about these. It is hoped that such remarks will have the additional advantage that mention will have been made in this and the preceding paper of all the principal aspects of our present knowledge about free convection in air from flat plates at uniform surface temperature.

2. The laminar flow

An extensive study of the laminar free convective boundary layer on a heated vertical plate in air was made by Schmidt & Beckmann (1930). Along with their own experiments they report a numerical computation of the velocity and temperature profiles carried out by Pohlhausen. Pohlhausen was obliged by poor convergence to fit to experimental values of the velocity and temperature gradients instead of the theoretical boundary conditions at the wall. However, Ostrach (1953) has repeated the work by a better method using the proper conditions and obtained closely similar profiles. Agreement between theory and experiment is good, and this flow is well understood. The results have, incidentally been extended to Prandtl numbers other than that of air by the work of Saunders (1939), Squire (1953, § 20) and Ostrach (1953).

This information may be applied to an inclined plate simply by replacing g by $g \cos \alpha$. In the boundary-layer approximation the momentum equation normal to the plate reduces to a balance between the buoyancy force and a pressure gradient. The term produced in the longitudinal momentum equation by these pressure variations is small in the boundary-layer approximation. The only difference from the vertical plate case is that the neglected terms are of order $(\delta/L) \tan \alpha$ (where δ and L are the transverse and longitudinal length scales) instead of $(\delta/L)^2$ smaller than the retained terms. This is of no importance with the Grashof numbers occurring in practice.

One implication of this theory is that the boundary-layer thickness (at fixed Gr) increases only as $\sec^{\frac{1}{4}} \alpha$ (which is less than 2 even when α is as large as 80°). It is thus consistent that the flow retains its boundary-layer characteristics.

No detailed experimental check of these conclusions has been made, but a few measurements of the laminar flow that I have made are quite consistent with them. There is also some evidence on the temperature field in the work of Rich (1953).

For use later in this paper, we note the appropriate similarity parameters for this flow

$$\eta = \left[\frac{g \cos \alpha (T_1 - T_0)}{4\nu^2 x T_0}\right]^{\frac{1}{4}} y,$$

$$\zeta = \left[\frac{T_0}{64 g \cos \alpha \nu^2 (T_1 - T_0)}\right]^{\frac{1}{4}} \psi,$$

ψ being the stream function such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$, and

$$\theta = (T - T_0)/(T_1 - T_0).$$

Pohlhausen's and Ostrach's solutions give tables of ζ and θ and their derivatives as functions of η.

The sign of α does not affect the solutions. The laminar flows above and below the plate at the same angle are alike. But the extent of the flow that is laminar can be different above and below, through the stabilizing or destabilizing action of the stratification.

3. The fully turbulent flow

Experience of turbulent flows (Townsend 1956) leads us to expect that at large enough Grashof numbers the motion is independent of the conditions further upstream in which the flow has become turbulent. The interest of this for transition investigations arises from the indications (Dhawan & Narasimha 1958) that even in a region of spreading intermittent turbulence one finds this characteristic structure.

Our knowledge of this structure in free convective flows is not very great. For the case of a vertical plate, it derives largely from the work of Griffiths & Davis (1922), who investigated the heat transfer and the mean velocity and temperature fields. Unfortunately, their anemometer and thermometer were not designed for averaging and present-day experience makes it doubtful whether they could obtain a true mean by eye. To my knowledge no measurements have been made of any of the other quantities that characterize turbulent flows. There is thus need for more experimental study of this flow, though it is not clear how the instrumentation difficulties can be overcome.

One point on which there is further data is the heat transfer, which, according to measurements by Griffiths & Davis (1922), Saunders (1936, 1939) and Michev (1947) is constant up the plate. This implies that the temperature gradient at the wall is constant. On the other hand, the velocity and temperature cannot both have profiles of constant shape and length scales; if they did the heat lost from the wall could not be carried off by the flow (in other words, the heat integral equation could not be satisfied). Hence, if the observations of constant heat transfer are correct, the question arises why the turbulence adjusts itself to produce this when it cannot make all properties constant. These considerations suggest perhaps that the boundary layer has a double (inner and outer) structure like that of forced-flow turbulent boundary layers.

Eckert & Jackson (1951) and Bayley (1955) have given different theoretical analyses of turbulent free convection from vertical plates. They were both forced to make extensive assumptions (such as the form of the profiles) and to carry over without apparent justification certain relationships (such as the dependence of wall stress and heat transfer on boundary-layer thickness) from forced convection. These theories are, therefore, of no help in understanding the basic mechanics of the flow; nor can one have great confidence in using the results for particular calculations, though they may serve their intended purpose of providing heat-transfer formulae for engineering use. For the present purpose, they may also usefully be regarded as providing algebraic representations of Griffiths & Davis's data.

When the plate is inclined, the structure of the turbulence will be altered by the normal buoyancy forces. The way in which this happens is a matter of considerable interest. However, I know of no experiments on turbulent flow on inclined plates other than those at large negative α reported in Tritton (1963*b*).

4. Approach to investigation of transition

In an investigation of how the transition to turbulence moves up and down the plate as the angle is varied, the choice of features examined is to some extent arbitrary. I choose to pay particular attention to turbulent spots—the localized bursts of turbulence whose spread into surrounding laminar fluid seems (as was first suggested by Emmons 1951) to be an important stage of the transition process in boundary layers. One reason for this was experimental ease, but such an emphasis was also indicated by the behaviour of forced flow along a wall (Tritton 1960, § 10.1). The line of thought will not be sketched here as it is now somewhat out-of-date, though there is nothing in the recent developments that makes the choice unfortunate.

Dhawan & Narasimha (1958) found that in forced boundary layers the position at which spots first appear is distinctive. A preliminary experiment with the plate vertical suggested that this was also true in my experiment. It indicated too that there was a distinctive position at which the intermittency factor reached unity. (Theoretically it asymptotes to 1, but this supposes that the changes, as a spot passes, between wholly turbulent motion and wholly laminar occurs instantaneously. In fact it must take a finite time and when the duration of the laminar spells becomes as small as this the velocity will never settle down to its characteristic laminar value.) I thus though it the best policy to regard these two positions as demarcating the transition.

The variation with angle of the first position might, broadly speaking, be regarded as indicating the effect of stratification on the stability of the laminar flow, though the growth of disturbances must start somewhere upstream of it. Rather more definitely, the variation of the second position indicates how stratification influences the development of spots.

5. Experimental procedure

The heated plate has been described earlier (Tritton 1963*b*, § 3). There was only one difference from that arrangement. If the lower edge had been left ungarnished, there would have been an accumulation of hot static air just below it; it would then have been difficult to know just where the effective position of $x = 0$ was. To overcome this, the plate was fitted with a bleed consisting of a slit a few millimetres wide running the width of the plate, through which air was gently sucked by a water-suction pump. It had, of course, to be tested that the sucking was not so strong as to draw air down the plate; this was done with ammonium chloride smoke. This bleed was used throughout the transition investigation. For these experiments, the upper edge could safely be assumed to have no influence and conditions there did not have to be regulated.

The temperature loading was approximately the same, in the range 30–40 °C above ambient, during surveys at different angles. The results can, and will, be presented in non-dimensional form, but the comparisons are probably better at similar $T_1 - T_0$ (for reasons mentioned later in this section). I occasionally went to a higher loading, sometimes just to see the effect of changes and sometimes to attain higher Grashof numbers. The heating voltages were always adjusted to make the plate temperature uniform.

All surveying was done on the centre line in the z-direction. Quartz fibre anemometers (Tritton 1963*a*) were used for the surveying; indeed this is an application to which they are quite well suited.

The method was to traverse a fibre anemometer in the x-direction and observe whether (i) there were only the weak waverings that occurred whether or not the fibre was in the boundary layer, or (ii) there were laminar periods (perhaps again with some wavering due to room disturbances) interspersed with sharply contrasted periods of fibre activity, or (iii) the fibre was in motion all the time with no laminar spells detectable. (I did *not* consider laminar spells to exist simply when the fibre end was occasionally particularly still; I required that all the still periods should occur at about the same deflexion, so that a characteristic velocity could be assigned to the laminar spells. I found this criterion a practicable one even when the intermittency factor was very close to unity.)

The positions thus assigned to the two developments undoubtedly have some dependence on the characteristics of the fibre used, and possibly also on the observer. However, when the ranges of x over which the designation of the flow ((i), (ii), or (iii) above) was doubtful were short, it is likely that the results were a good indication of the actual starts of intermittent turbulence and full turbulence. The one obvious qualification of this is that laminar zones small compared with the length of the fibre would not be detected. The main purpose of the work being comparison of the behaviour at different angles, the same fibre was used throughout (thus making it also advantageous to have a constant temperature loading). Its length was 3·07 cm. The other quantity that might influence the results is presumably ν_n/ϕ_n^2 (Tritton 1959); this was 8·94 sec^{-1} ($\nu_1 = 31{\cdot}5$ c/s). The smallest fluctuation in h readily observed was around 0·002 cm (about half the fibre diameter); this corresponds to a velocity change of about 1·5 cm sec^{-1}.

It would, of course, not have been satisfactory to make the traverses in the x-direction without attention to the values of y. My observations at each x were made during a traverse across the boundary layer from close to the wall to well outside the point of inflexion in the velocity profile (as indicated by the laminar solution); a typical range would be $0{\cdot}1 < y < 1{\cdot}5$ cm. The number of y-settings varied, being small when the fibre was obviously nowhere near either development and there were no other points of interest to investigate, but averaged around 6. The range of x over which different designations seemed appropriate at different y was always short; thus the variations across the boundary layer did not upset the general approach of assigning values of x to the developments of the flow.

6. Experimental results; introduction

My original aim was to show the effect of stratification on the transition by plotting graphs of $Gr^{\frac{1}{4}}$ corresponding to just non-zero and unit intermittency factor against α. This has been only partially achieved; some of the observations, though manifest, did not prove repeatable. The situation is probably best explained by a brief history of the experiment.

I started with the plate vertical and changed α in stages of about 12°, surveying below and above at each inclination. Seemingly quite satisfactory values of x for the two developments were obtained for the vertical plate and for the first three positive values of α (stable stratification).

In particular, the spots at their first appearance could, except when there was a lot of movement in the room, be clearly distinguished from any extraneous waverings. Probably the feature that best indicated whether the transition should be sought at higher or lower x was the higher frequencies in the spots, but closer inspection revealed a characteristic sequence of developments that lent stronger support to my interpretation of the fibre's behaviour. As x increased, the motion associated with the spots first appeared, with a low intermittency factor, over a restricted range of y; it then spread to the whole boundary layer; and above this there was an increase of both its amplitude† and intermittency factor, quickly producing a marked contrast with the behaviour below the

† This does not necessarily imply that the intensity of turbulence within a spot was increasing; it is quite likely due to the air being turbulent over a greater proportion of the fibre's length.

transition. Just below the value of x having occasional turbulence for some y, there could be found a very short range of y ($\sim 0 \cdot 1$ cm) in which the fibre occasionally moved jerkily as if the flow temporarily changed mode, though no real turbulence was detected. The whole development just described occupied a range of x small compared with x itself.†

But, when a search was made for a similar development at the fourth angle, it was unsuccessful. Moreover, I then returned to some of the previous angles, including 0, and was no longer able to find it at these. The behaviour at large x was not noticeably different from before. Close to the lower edge there was a distinct increase in the fluctuations, but probably the main cause of the difficulty was that the change between these two positions was now gradual and no distinct value of x could be assigned to it. A number of experiments in which I took care to make as little disturbance as possible and carried out early in the morning so that there had been no movement in the room for some hours, failed to reproduce the earlier behaviour.

There can be little doubt that a genuine change in the origin of the turbulence had occurred. Since the different behaviours were not producible at will, it is difficult to infer the cause, but I do not think it was a change in the heated plate apparatus; alterations to the features that could change unnoticed (such as temperature loading, strength of the bleed, entry conditions, roughness due to dirt) had no effect. It is probable, as was concluded for the flow described in the preceding paper, that the general circulation of air in the room was not wholly governed by the hot plate; I suspect that the change was basically a change in this that gave rise to some new instability, probably producing intermittent turbulence in the circulating air. All that was then necessary for the development of the boundary-layer turbulence was amplification by the shear, whereas, without this instability, the appearance of spots in a manner similar to that in forced flow was a necessary preliminary. Possibly the laminar free convection is unstable right from $x = 0$ to some *finite* disturbances.

Despite this complication the surveys were not without their positive results—the variation of the first position of full turbulence was of particular interest—and these will be subjects of subsequent sections.

This account of the course of the experiments has so far made mention only of the flow below the plate. The concurrent investigation of that above did not show any sudden change in behaviour, but the previous observations were such that one could have occurred unnoticed. The first appearance of intermittent turbulence moved, as the inclination was increased, to a lower value of x (see § 13) and had already reached the lower edge before the change in the flow below occurred. I think this was a genuine trend resulting from thermal instability, but one effect was that the flow above before the change was not so different from that either above or below after.

† For example, at $\alpha = 13 \cdot 9^\circ$ and $(T_1 - T_0)/T_0 = 0 \cdot 125$, the observations were as follows: $x = 17 \cdot 5$ cm—boundary layer completely laminar; at $19 \cdot 7$ cm—jerky motion around $y = 0 \cdot 8$ cm, but laminar elsewhere; at $20 \cdot 8$ cm—intermittent turbulence when $0 \cdot 55 < y < 0 \cdot 95$ cm, but laminar elsewhere; at $23 \cdot 8$ cm—intermittent turbulence throughout boundary layer.

7. Grashof numbers of transition

In view of their non-repeatability, the significance of the Gr values determined for the first appearance of spots is uncertain, but it is possible that the position would be similar whenever the flow developed in this way. Hence, the results obtained before the change in behaviour are presented in figure 1. (In this and

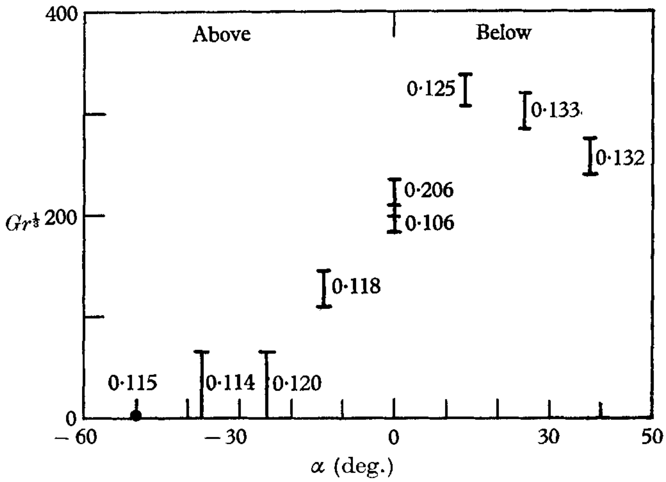

FIGURE 1. Grashof numbers of first appearance of fluctuations. Points are labelled with values of $(T_1 - T_0)/T_0$.

subsequent figures, the ordinate is $Gr^{\frac{1}{3}}$; this is proportional to the distance from the lower edge and so its use gives a clearer impression of the physical behaviour than Gr.) At each angle a short range of $Gr^{\frac{1}{3}}$ is marked on the graph; this is the range over which the designation of the flow (as described in § 5) was uncertain or varied with y. The numbers against the points are values of $(T_1 - T_0)/T_0$. The good agreement for the two different loadings at $\alpha = 0$ encourages the view that the results are of some significance despite the complications.

Figure 2 shows values of $Gr^{\frac{1}{3}}$ at which laminar spells were no longer observable, plotted on the same principles. These values were determined during the same series of runs as those of figure 1, so that, except for $\alpha = 49{\cdot}6^\circ$, all the positive α results correspond to intermittent turbulence starting at a finite value of x. However, since $Gr^{\frac{1}{3}}$ for the first full turbulence is throughout considerably larger than that for the first intermittent turbulence, figure 2 would probably be only a little altered if the results had been obtained after the change in behaviour. At $\alpha = 49{\cdot}6^\circ$ the temperature loading was increased to the maximum possible to reach the high Grashof numbers involved; even so full turbulence was only just beginning at some value of y at the top of the plate. The results do not extend to larger α than $49{\cdot}6^\circ$, in the case of figure 1 because of the change in behaviour and in that of figure 2 because the plate was not long enough.

Though $Gr^{\frac{1}{3}}$ is the physically most significant parameter, a more immediate picture of how the flow varies up the plate at the various α is given by having

$$\left(\frac{g}{\nu^2}\frac{T_1 - T_0}{T_0}\right)^{\frac{1}{3}} x = (Gr \sec\alpha)^{\frac{1}{3}}$$

as ordinate. Figure 3 shows the data of both figures 1 and 2 in this way.

It is clear that the transition region is throughout so long that a general practical list of the types of flow must be laminar, transitional and turbulent, rather than just the first and last (though in the flow *above* the plate it may be preferable to have just two classes, developing turbulence and full turbulence—

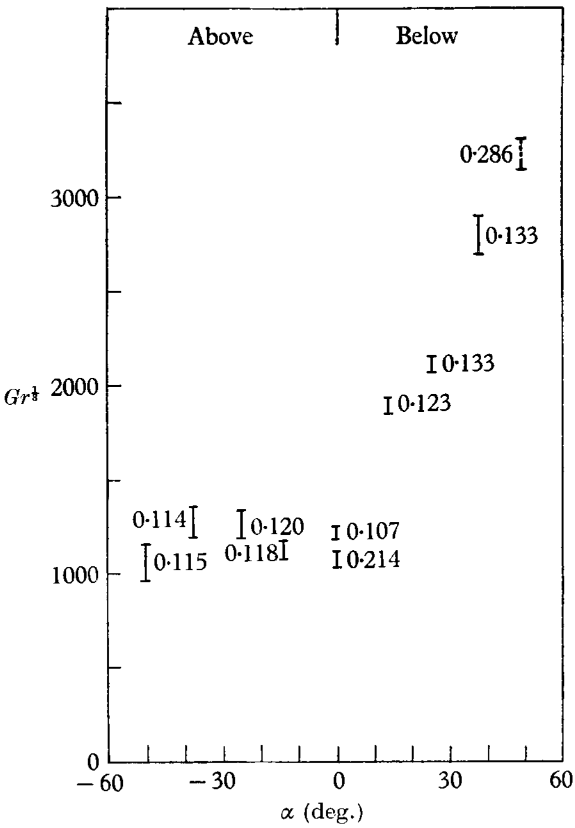

FIGURE 2. Grashof numbers of first full turbulence. Points are labelled with values of $(T_1 - T_0)/T_0$.

see § 13). This is true also of forced boundary layers, but not to the same extent. In free convection there may be cases of practical importance where the principal feature is a boundary layer in which turbulence is intermittent with laminar spells.

8. Distance from wall of first instability

It has been remarked that when the first fluctuations occurred some distance up the plate their very first appearance was over a short y-range. The relationship of this to the laminar velocity profile is worth recording as extra information about the instability: it was throughout a little further from the wall than the point of inflexion. Values of η (notation of § 2) of the first fluctuations varied between 1·8 and 2·5 with no systematic dependence on α, while the point of inflexion on Pohlhausen's and Ostrach's solutions is at $\eta = 1{\cdot}85$.

9. Transition on a vertical plate

The remarks in § 7 about the length of the transition region apply even when the plate is vertical. The Richardson number is then zero and so the effect of buoyancy forces on the development of turbulence is small;† the contrast with forced boundary layers is more significant than when there is stratification.

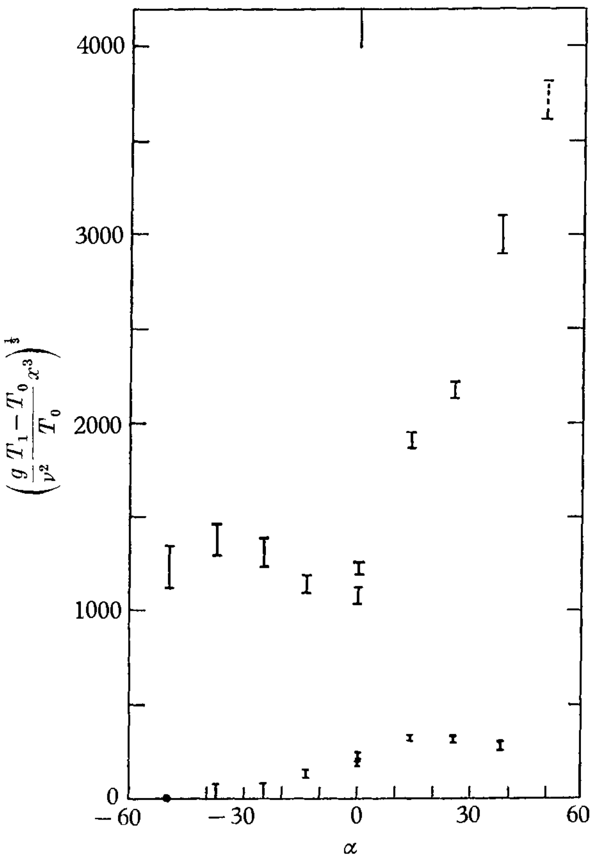

FIGURE 3. The data of figures 1 and 2 re-presented to show more directly the flow regimes at various α.

This may be a consequence of the Reynolds number in the transition region being low. Dhawan & Narasimha's (1958) results for forced boundary layers indicate that the lower the Reynolds number, the longer is the transition region relative to the length of the preceding laminar region. This information is in the form of an empirical relationship between Reynolds numbers based on these two lengths; a comparison of the free convection data with this would not be meaningful because of the variation of the velocity scale with x. We can just note that the long transition region may be in accord with the trend of the forced flow.

† An observation that suggests that it is not completely non-existent will be described in § 12.

The low Reynolds number is in turn doubtless a consequence of the point of inflexion in the velocity profile (Rayleigh 1880; Lin 1945). Just how large an effect this has is probably indicated by the local Reynolds number. Basing this on the maximum velocity and the distance from the wall at which the velocity falls to 0·01 times its maximum, Pohlhausen's and Ostrach's solutions give

$$R = 4{\cdot}5Gr^{\frac{1}{4}}.$$

This is about 250 at $Gr^{\frac{1}{3}} = 210$, a factor of about 20 smaller than the roughly analogous quantity for Blasius flow.

Whatever the mechanism underlying the long transition region, it means that the practice, sometimes adopted, of writing down a single critical Grashof number for vertical plate convection is unsatisfactory. It undoubtedly explains why there is considerable divergence in the literature amongst the Grashof numbers of the onset of turbulence indicated in different ways. The following is a list of $Gr^{\frac{1}{3}}$ values either given by other authors or calculable from their data, and to be compared with my figures of 210 (0 after the change in behaviour described in § 6) and 1150 for the start of intermittent and full turbulence respectively:

Griffiths & Davis (1922)—transition region over range $500 < Gr^{\frac{1}{3}} < 1400$ or $350 < Gr^{\frac{1}{3}} < 950$ (two separate experiments) indicated by variations in heat transfer.

Schmidt & Beckmann (1930)—(quartz-fibre observations). Laminar flow at $Gr^{\frac{1}{3}} = 170$; intermittent turbulence at $Gr^{\frac{1}{3}} = 350$ and 520; intermittent or full turbulence at $Gr^{\frac{1}{3}} = 700$.

Schmidt (1932)—Schlieren photographs show laminar flow occurring over whole plate with maximum $Gr^{\frac{1}{3}}$ of 1000.

Hermann (1936)—(Schlieren photography specifically to find transition and interpreted on supposition that there is always a definite instantaneous critical Grashof number but that this fluctuates). 'Time mean value of critical position' determined in four separate experiments, giving $Gr^{\frac{1}{3}}$ as 1100, 880, 1090 and 930.

Saunders (1936)—transition region over range $1000 < Gr^{\frac{1}{3}} < 2500$ indicated by variations in heat transfer.

Rich (1953)—(interferometric study of temperature field). Range covered was $150 < Gr^{\frac{1}{3}} < 750$, and throughout there was a systematic deviation from Pohlhausen's theory (see § 2) which Rich attributed to flow being transitional; the heat transfer, on the other hand, corresponded throughout to laminar flow.

Eckert, Soehngen & Schneider (1955)—(interferometric study of temperature field; also smoke experiments). Turbulence observed to develop from Tollmien–Schlichting waves starting at $Gr^{\frac{1}{3}} = 750$.

Only experiments at the Prandtl number of air have been included in the above list. Of these various results, Eckert, Soehngen & Schneider's is the most difficult to reconcile with my own. That a motion resembling Tollmien–Schlichting waves can occur in this type of flow is confirmed by experiments in other fluids by Fujii (1959). This in itself does not contradict my results, since my method of investigation was not designed to detect Tollmien–Schlichting waves, but one would expect them lower down the plate than the first appearance of spots.

A possible explanation is that unlike forced flow (Narasimha 1957; Dhawan & Narashima 1958) the spots originate over a large range of x, and that Eckert, Soehngen & Schneider observed waves preceding some of the later spots. (They say that they had to wait for the room to be particularly still and possibly they also rejected genuine boundary-layer instabilities of lower origin.) This would provide an explanation, alternative or additional to that considered earlier, of the long transition region.

It may be, on the other hand, that a much longer wholly laminar regime than I have observed *can* occur when the room disturbances are weak enough. (Saunders also suggests that draughts were responsible for Griffiths & Davis's heat transfer measurements not agreeing with his own.) If so, the implication is *not* that the fluctuations I observed would have been there even without the convective flow, but that the point at which amplification by the flow begins is strongly dependent on the disturbance level.

10. Effect of stable stratification

The behaviour when α is increased from 0 is probably the most interesting and successful part of the experiment. However, the main results have been indicated in figures 1–3, and, apart from one additional observation, this section will consist of comments on them.

Again, of course, the worth of the data on the first appearance of spots is open to question. However, it does seem that $Gr^{\frac{1}{3}}$ remains of the same order and thus that the stability of the laminar flow is not greatly increased by the stratification. In view of the unsatisfactory state of the stability theory for this flow (Plapp 1957), there is no very sure way of knowing whether this accords with theory. However, a consideration of Richardson numbers may give us some idea.

In general terms, the Richardson number is an indication of the importance of the normal buoyancy force as compared with that of the shear. It enters into any consideration of the stability of stratified flows. However, the information on its effect is restricted to rather few cases and yet is not readily summarized. With considerable over-simplification, one might say that for an inviscid fluid (Chandrasekhar 1961, §§ 102–4) stratification prevents instability when the Richardson number exceeds $\frac{1}{4}$. Schlichting's (1935) theory for forced boundary layers indicates that values very much less than 0·25 produce a significant increase in the critical Reynolds number.

The Richardson number can be defined either as an overall parameter for each station on the plate or as a local parameter depending also on the distance from the plate. The two forms are

$$Ri_0 = g \sin\alpha \left(\frac{T_1 - T_0}{T_0}\right)\frac{\delta}{u^2},$$

and

$$Ri_l = -\frac{g \sin\alpha (\partial T/\partial y)}{T_0 (\partial u/\partial y)^2}.$$

The profiles taken in the various theories mentioned above were such that the difference between Ri_0 and Ri_l was unimportant. This is not the case for the present flow, and it is not obvious what is the best quantity to use in comparisons.

The information on the laminar flow summarized in § 2 indicates that

$$Ri_0 = 13{\cdot}2 \tan\alpha / Gr^{\frac{1}{4}}$$

(if u is taken as the maximum velocity and δ as the value of y at which the velocity has fallen to 1 % of its maximum) and

$$Ri_l = 0{\cdot}177\theta' \tan\alpha / \zeta''^2 Gr^{\frac{1}{4}}.$$

Ri_l varies greatly across the boundary layer (having a singularity at the velocity maximum); at the point of inflexion

$$Ri_{li} = 2{\cdot}3 \tan\alpha / Gr^{\frac{1}{4}}.$$

The data shown in figure 1 (for positive α) give $Gr^{\frac{1}{4}}$ in the range 50 to 75 and have $\tan\alpha$ going up to 0·78. We are thus concerned with Ri_0 in the range 0 to about 0·18 and Ri_{li} in the range 0 to about 0·03. These values are sufficiently less than 0·25 that it is possible for the stratification to have little effect. On the other hand, Schlichting's (1935) theory shows such values giving a significant increase in the critical Reynolds number in forced boundary layers. Hence, the rough constancy of critical $Gr^{\frac{1}{4}}$ may perhaps be interpreted as indicating that the effect of the point of inflexion remains strong even when there is stable stratification.

$Gr^{\frac{1}{4}}$	Ri_{li} (E. & J.)	Ri_{li} (B.)
300	$0{\cdot}034 \tan\alpha$	$0{\cdot}048 \tan\alpha$
1000	$0{\cdot}024 \tan\alpha$	$0{\cdot}056 \tan\alpha$
3000	$0{\cdot}017 \tan\alpha$	$0{\cdot}058 \tan\alpha$

TABLE 1. Local Richardson numbers from the profiles of Eckert & Jackson and Bayley.

It is when the spots have appeared that the stratification has its important effect. The possibility of a flow being almost entirely in a state of intermittent turbulence was demonstrated strikingly as a result of the rapid increase of $Gr^{\frac{1}{4}}$ corresponding to first full turbulence. It seems quite likely that as α becomes large the flow can never become fully turbulent even in the ideal case of a semi-infinite plate—that is to say the intermittency factor may asymptote to a value less than unity. For such a conclusion to be established with conviction an apparatus capable of much larger Grashof numbers than the present set-up would be needed.

The reduced rate of spot growth is in accord with the action of stabilizing density gradients on turbulence as revealed by the work of Nichol (reported in Townsend, 1958), Ellison (1957), Townsend (1958), and Ellison & Turner (1959, 1960). Again an estimate of the Richardson number—this time that of the turbulent flow—can put the comparison on a more quantitative basis. There are no data on the profiles of the inclined plate turbulent flow. In the absence of any more satisfactory procedure, a value for the Richardson number may be calculated from Eckert & Jackson's and Bayley's profiles for a vertical plate (see § 3), supposing (doubtless incorrectly) that they are unchanged by the inclination. This gives the values for Ri_l at the point of inflexion shown in table 1.

The reduced rate of spot growth was observed as $\tan\alpha$ was increased from 0 to about 1·2. Clearly the actual figures for Ri_{lt} have no significance, but it may be hoped that the order of magnitude is correctly indicated. The observations confirm the conclusion reached in the papers mentioned above that stable stratification has a strong action on the turbulence at Richardson numbers very much less than the value unity originally suggested by Richardson (1920) as the limit of possible turbulent motion.

Comparison with the physical processes considered in these papers suggests two ways in which the stratification may be operating to lower the rate of increase of intermittency factor. Firstly, the laminar flow may be unstable, but the turbulence is unable to maintain itself (because of modifications to Richardson's energy-balance argument resulting from transport of energy, dissipation and inequality of eddy viscosity and eddy conductivity—again an over-simplified summary of a complex state of affairs). The result would be a continual formation of spots in which the turbulence then tends to die out again.

Secondly, Ellison & Turner (1959) have shown that an increase in Richardson number leads to a decrease in the rate at which a turbulent flow entrains ambient fluid. This effect will be present in the free convection on an inclined plate. However, a spotty flow develops to full turbulence by entrainment not so much of fluid outside the boundary layer as of laminar fluid in the boundary layer (i.e. at different x and z). We have to consider whether stratification could inhibit this too. Probably it does, because the turbulent temperature profile almost certainly has greater potential energy (for unit area of plate) than the laminar; that is to say it gives a larger value of

$$g\sin\alpha\int_0^\infty \frac{\overline{T}-T_0}{T_0}\,y\,dy.$$

Hence, an increase of intermittency factor involves work being done in a way that is not involved in the vertical plate flow. The information does not exist for this to be established with certainty, but it is interesting to compare

$$I = \int_0^\infty \frac{\overline{T}-T_0}{T_0}\,y\,dy$$

estimated for the laminar and turbulent vertical plate flows, again with Eckert & Jackson's profile for the latter. Then†

$$\frac{I_{\text{turb.}}}{I_{\text{lam.}}} = 0{\cdot}0066\,Gr^{\frac{3}{10}}$$

which is greater than unity when $Gr^{\frac{1}{4}} > 270$. Moreover, this probably underestimates $I_{\text{turb.}}/I_{\text{lam.}}$, as Eckert & Jackson's profile has $\overline{T}-T_0$ going to zero at a finite y, and the tail is likely to make an important contribution to $I_{\text{turb.}}$.

† $I_{\text{lam.}}$ was calculated by taking

$$\theta = \operatorname{erfc}(0{\cdot}452\eta) \quad \text{(notation of § 2)}$$

which is a good approximation to Pohlhausen's and Ostrach's profiles, useful when an algebraic form is required.

One additional observation, made incidentally during the course of the experiments, may further illuminate the position, though it is not so far placed on a rigorous quantitative basis. The time-scale of the jumps between laminar and turbulent flow increases as α increases. Little difference was observed around the first appearance of spots, but at positions of similar intermittency factor around $\frac{1}{2}$ (and temperature loading throughout about 35 °C) the typical duration of a single laminar or turbulent spell increased from 1 sec or less at $\alpha = 0$ to 5 sec or 10 sec at $\alpha = 49{\cdot}6°$. This suggests that the increasing Richardson number leads to some spots dying right out rather than the growth of all of them being equally inhibited. It also implies, unfortunately, that the length-scale of a single spot was large compared with the width of the plate, so the approximation to a flow infinite in the z-direction may not be satisfactory.

11. Glacier winds

There is a meteorological flow quite closely analogous to the present laboratory study, namely the glacier wind (Tollner 1931). Ekhart (1934) observed the motion of tethered balloons in a glacier wind and in consequence remarked that this is a particularly turbulent kind of wind. In view of the strong stable stratification this is just the reverse of what one would expect. An explanation of Ekhart's observations can now be advanced. The vigorous motion of his balloons may well have been produced not by the eddies of full turbulence, but by jumps in the velocity as the flow changed from laminar to turbulent and vice versa.

12. Phenomenon of two characteristic laminar velocities

During the observation of intermittent turbulence, it became apparent that the laminar spells sometimes had not one but two distinct characteristic velocities. This occurred for some values of x both when the plate was vertical and for all the positive values of α, but not in the flow above the plate. When it occurred, it did so right through the boundary layer.

When there was a single laminar velocity (still considering $\alpha \geqslant 0$), its relationship at various y to the fluctuations was in qualitative accord with the mean velocities of laminar and turbulent spells being those corresponding to wholly laminar and wholly turbulent flow. The mean velocity during turbulence was greater both very close to and far from the wall, but had a lower maximum in between.

The two velocity phenomenon was usually to be found at higher values of x. Then one velocity was still related as above to the mean turbulent velocity; the other was very much lower and indeed sometimes almost zero. Figure 4 shows velocities corresponding to positions at which the fibre repeatedly came to rest plotted against y for one α and x; a curve representing the wholly laminar flow profile is included. I did not obtain such quantitative data at other α and x, but the qualitative features were the same.

The occurrence of the two velocities seemed randomly distributed in time. The relative number of times each occurred was very variable, but further work is needed before any system can be seen in this variation. There were occasions when the low speed was more numerous by a factor of 5–10 right through the boundary layer. On the other hand I never observed, except at isolated values of y, spells at the low speed without any at the high, so that the problem of whether a low-speed spell should be regarded as a true laminar one for the purpose of defining intermittent turbulence (§ 5) did not arise.

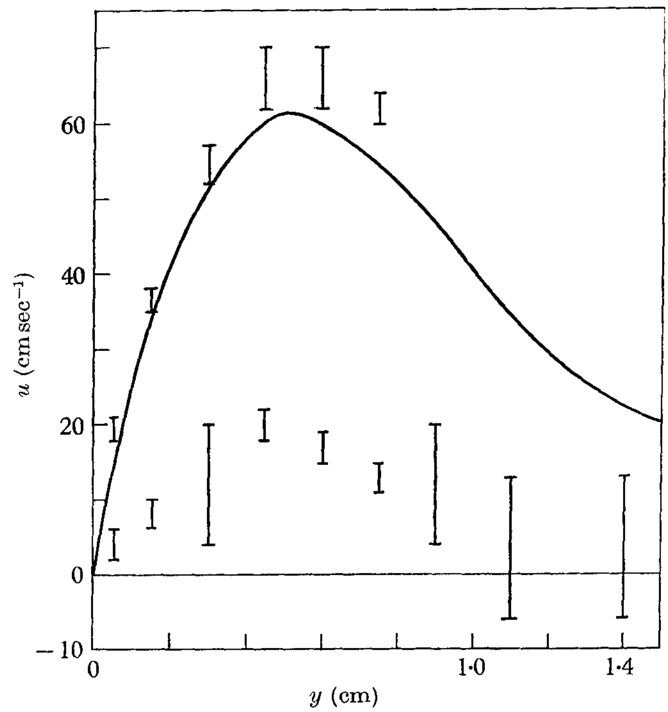

FIGURE 4. Velocities during laminar spells for $\alpha = 25{\cdot}3°$, $(T_1 - T_0)/T_0 = 0{\cdot}133$, $x = 104{\cdot}0$ cm. The continuous curve shows the wholly laminar solution for these conditions.

The origin of the low-speed spells is not apparent. In the stably stratified flows they could consist of fluid that has formerly been turbulent but, as a result of the processes considered in § 10, is so no longer; but this does not explain their occurrence in the vertical plate flow. Once they have occurred it is possible to see how blobs of almost stationary laminar fluid can remain in that condition. The turbulent spot in front of a blob is moving up the plate faster than the blob. Continuity therefore requires an inflow; this will consist of cold air with little momentum and so will itself have little tendency to move up the plate. Moreover, the inflow will be stably stratified—the turbulent air at larger x is hotter—which will tend to preserve the laminar state of the blob. This model can be regarded as a large eddy motion arising out of the turbulence but, because of the density variations, not generating smaller eddies.

If this mechanism is correct it carries the implication that the density variations affect the development of turbulence even in the vertical plate case.

The non-occurrence of the phenomenon in the negative α flows presumably means that the conditions are not right for the blobs to originate; if they did the suggested mechanism for their preservation would still work.

13. Effect of unstable stratification

The experiments at negative α did not produce results of such interest. However, a few comments should be made about the results shown in figures 1–3 and other incidental observations.

The flow does, of course, tend towards that described in the preceding paper (Tritton 1963*b*). It is, therefore, not to be expected that, as $-\alpha$ becomes large, the laminar spells should retain their clear-cut identity. At low values of x, wholly laminar flow will be observed except when there is a large 'eddy' (or 'plume') at the point of observation. As x increases small-scale turbulence will gradually develop around the large eddies. The greater difficulty in deciding where full turbulence started (as indicated by the longer ranges in figure 2) is thus understandable.

The first appearance of fluctuations is correspondingly to be expected to go down to $x = 0$. As figure 1 shows, there was one negative value of α at which a non-zero x could be assigned to this and two at which it was not clear whether the fluctuations began right at the lower edge but certain that they did at a low $Gr^{\frac{1}{3}}$. In these two cases, smoke experiments showed that there was an intermittent billowing motion in which air close to the lower edge moved vertically instead of up the plate. However, it then got mixed with the inflow to the boundary layer at a larger x and so was spasmodically bent over and carried into the layer, thus producing intermittent turbulence. The whole process occurred within a few centimetres of the lower edge. At $\alpha = -50{\cdot}05°$, there was intermittent turbulence occurring, without question, right at $x = 0$; as, however, the flow below was by this stage of the experiment showing the same, the significance of this is doubtful.

Throughout the work the behaviour of the fibre indicated two differences of the unstably stratified flow from the stably stratified:

(i) The maximum (with respect to y) laminar velocity is not clearly greater than the maximum of the mean during turbulence. (This was true even at $\alpha = -13{\cdot}9°$.)

(ii) Comparing at similar inclinations and Grashof numbers, the frequencies involved in the turbulence are conspicuously lower.

These two points, together with the fact that laminar spells occur up to values of $Gr^{\frac{1}{3}}$ as large as the maximum at which they were observed with the plate vertical (see figure 2), suggest that once fluctuations have appeared as a result of thermal instability there is little tendency for developments like those on a vertical plate to be superimposed.

Since this paper was prepared, a closely related paper by Szewczyk (1962) has appeared. A consistent physical picture can be obtained by taking Szewczyk's observations as a description of the earlier stages of transition and those in the present paper as a description of the later stages. The difference in Prandtl number prevents a detailed quantitative comparison, but the high Grashof numbers of transition quoted by Szewczyk perhaps indicate that, in my experiments, disturbances in the room were causing early transition.

On the other hand, in a still more recent paper, Kurtz & Crandall (1962) calculate a critical Reynolds number, for the stability of the laminar convection on a vertical plate with the Prandtl number of air, that corresponds to $Gr^{\frac{1}{4}} = 120$.

This work was done while I was a research student at the Cavendish Laboratory, Cambridge. I wish to thank Dr A. A. Townsend, Mr W. E. Thompson, and the Department of Scientific and Industrial Research for supplying my academic, technical, and material needs.

REFERENCES

Bayley, F. J. 1955 *Proc. Inst. Mech. Engrs*, **169**, 361.

Chandrasekhar, S. 1961 *Hydrodynamic and Hydromagnetic Stability*. Oxford University Press.

Dhawan, S. & Narasimha, R. 1958 *J. Fluid Mech.* **3**, 418.

Eckert, E. R. G. & Jackson, T. W. 1951 *Nat. Adv. Comm. Aero., Wash., Tech. Rep.* no. 1015.

Eckert, E. R. G., Soehngen, E. & Schneider, P. J. 1955 50 *Jahre Grenzschichtforschung*, p. 407 (ed. by Görtler & Tollmein). Vieweg.

Ekhart, E. 1934 *Beitr. Phys. frei. Atmos.* **21**, 245.

Ellison, T. H. 1957 *J. Fluid Mech.* **2**, 456.

Ellison, T. H. & Turner, J. S. 1959 *J. Fluid Mech.* **6**, 423.

Ellison, T. H. & Turner, J. S. 1960 *J. Fluid Mech.* **8**, 529.

Emmons, H. W. 1951 *J. Aero. Sci.* **18**, 490.

Fujii, T. 1959 *Bull. Japan Soc. Mech. Engrs*, **2**, 551.

Griffiths, E. & Davis, A. H. 1922 *Special Rep.* no. 9, Food Investigation Board, D.S.I.R.

Hermann, R. 1936 *VDI Forschungsheft* 379 (translated as *Nat. Adv. Comm. Aero.*, Wash., Tech. Mem. no. 1366).

Kraus, W. 1955 *Messungen des Temperatur- und Geschwindigkeitsfeldes bei freier Konvektion*. Karlsruhe: Braun.

Kurtz, E. F. & Crandall, S. H. 1962 *J. Math. Phys.* **41**, 264.

Lin, C. C. 1945 *Quart. Appl. Math.* **3**, 218.

Michev, M. A. 1947 *Izv. Akad. Nauk. SSSR, Otd. tekh. Nauk.* no. 10.

Narasimha, R. 1957 *J. Aero. Sci.* **24**, 711.

Ostrach, S. 1953 *Nat. Adv. Comm. Aero., Wash., Tech. Rep.* no. 1111.

Plapp, J. E. 1957 *J. Aero. Sci.* **24**, 318.

Prandtl, L. 1952 *Essentials of Fluid Dynamics*. London: Blackie.

Rayleigh, Lord 1880 *Proc. Lond. Math. Soc.* **11**, 57.

Rich, B. R. 1953 *Trans. Amer. Soc. Mech. Engrs*, **75**, 489.

Richardson, L. F. 1920 *Proc. Roy. Soc.* A, **97**, 354.

Saunders, O. A. 1936 *Proc. Roy. Soc.* A, **157**, 278.

Saunders, O. A. 1939 *Proc. Roy. Soc.* A, **172**, 55.

Schlichting, H. 1935 *Z. angew. Math. Mech.* **15**, 313.

Schlichting, H. 1955 *Boundary Layer Theory*. London: Pergamon.

Schmidt, E. 1932 *Forsch. Geb. IngWes.* **3**, 181.

Schmidt, E. & Beckmann, W. 1930 *Tech. Mech. Thermodynam.* **1**, 341, 391.

Squire, H. B. 1953 Chap. XIV of *Modern Developments in Fluid Mechanics, High Speed Flow* (ed. by L. Howarth). Oxford University Press.

Szewczyk, A. A. 1962 *Internat. J. Heat Mass Transfer*, **5**, 903.

TOLLNER, H. 1931 *Met. Z.* **48**, 414.

TOWNSEND, A. A. 1956 *The Structure of Turbulent Shear Flow.* Cambridge University Press.

TOWNSEND, A. A. 1958 *J. Fluid Mech.* **3**, 361.

TRITTON, D. J. 1959 *J. Fluid Mech.* **6**, 547.

TRITTON, D. J. 1960 Ph.D. thesis, University of Cambridge.

TRITTON, D. J. 1963*a* *J. Fluid Mech.* **16**, 259.

TRITTON, D. J. 1963*b* *J. Fluid Mech.* **16**, 272.